Art-6 Laplace's Definite Integrals for $P_n(x)$

(i) Laplace's first Integral for $P_n(x)$:-

When n is +ve Integer,

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} \left[x \pm \sqrt{(x^2-1)} \cos\alpha \right]^n d\alpha$$

→ from Integrating calculus, we have

$$\int_0^{\pi} \frac{d\alpha}{a \pm b\cos\alpha} = \frac{\pi}{\sqrt{(a^2-b^2)}} , \text{ where } a^2 > b^2$$

Putting $a = 1-hx$ and $b = h\sqrt{x^2-1}$

So that
$$a^2 - b^2 = (1-hx)^2 - h^2(x^2-1)$$
$$= 1 + h^2x^2 - 2hx - h^2x^2 + h^2$$
$$= 1 - 2hx + h^2$$

We have

$$\pi(1-2xh+h^2) = \int_0^{\pi} \left[1 - hx \pm \sqrt{(x^2-1)}\cos\alpha \right]^{-1} d\alpha$$

$$= \int_0^{\pi} \left[1 - h\{x \mp \sqrt{x^2-1} \cos\alpha\}\right]^{-1} d\alpha$$

$$= \int_0^{\pi} [1 - ht]^{-1} d\alpha \quad \text{where } t = x \mp \sqrt{x^2-1} \cos\alpha$$

$$\pi \sum_{n=0}^{\infty} h^n P_n(x) = \int_0^{\pi} (1 + ht + h^2t^2 + \cdots + h^n t^n) d\alpha$$

Equating the coeff of h^n we have

$$\pi P_n(x) = \int_0^{\pi} t^n d\alpha = \int_0^{\pi} \left[x \mp \sqrt{(x^2-1)} \cos\phi\right]^n d\alpha$$

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} \left[x \pm \sqrt{(x^2-1)} \cos\phi\right]^n d\alpha$$

(ii) Laplace's Second Integral for $P_n(x)$:

When n is a positive Integer,

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} \frac{d\alpha}{\left[x \pm \sqrt{(x^2-1)}\cos\alpha\right]^{n+1}}$$

→ from Integral calculus,

$$\int_0^{\pi} \frac{d\alpha}{a \pm b\cos\alpha} = \frac{\pi}{\sqrt{a^2-b^2}}, \quad \text{where } a^2 > b^2$$

Putting $a = xh - 1$, $b = h\sqrt{x^2-1}$

so that $a^2 - b^2 = 1 - 2xh + h^2$

we have

$$\pi(1-2xh+h^2)^{-1/2} = \int_0^{\pi}\left[-1 + xh \pm h\sqrt{x^2-1}\cos\alpha\right]^{-1} d\alpha$$

$$\frac{\pi}{h}\left[1 - 2x\frac{1}{h} + \frac{1}{h^2}\right]^{-1/2} = \int_0^{\pi}\left[h\left\{x \pm \sqrt{(x^2-1)}\cos\alpha\right\} - 1\right]^{-1} d\alpha$$

$$\frac{\pi}{h}\sum_{n=0}^{\infty}\frac{1}{h^n}P_n(x) = \int_0^{\pi}(t-1)^{-1}\,d\alpha$$

where $t = h\left\{x \pm \sqrt{x^2-1}\cos\alpha\right\}$

$$= \int_0^{\pi}\frac{1}{t}\left[1-\frac{1}{t}\right]^{-1}d\alpha$$

$$= \int_0^{\pi}\frac{1}{t}\left[1+\frac{1}{t}+\frac{1}{t^2}+\cdots+\frac{1}{t^n}+\cdots\right]d\alpha$$

$$= \int_0^{\pi}\left[\frac{1}{t}+\frac{1}{t^2}+\frac{1}{t^3}+\cdots+\frac{1}{t^{n+1}}+\cdots\right]d\alpha$$

$$= \int_0^{\pi}\sum_{n=0}^{\infty}\frac{1}{t^{n+1}}\,d\alpha$$

$$= \sum_{n=0}^{\infty}\int_0^{\pi}\frac{1}{h^{n+1}\left\{x \pm \sqrt{(x^2-1)}\cos\alpha\right\}^{n+1}}$$

Art. — Orthogonality of $P_n(x)$

(i) $$\int_{-1}^{+1} P_m(x)\, P_n(x)\, dx = 0 \quad \text{if } m \neq n$$

(ii) $$\int_{-1}^{+1} [P_n(x)]^2\, dx = \frac{2}{2n+1} \quad \text{if } m = n$$

→ Legendre's equation may be written as —

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0$$

$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} + n(n+1)P_n = 0 \quad \text{—} (1)$$

$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\} + m(m+1)P_m = 0 \quad \text{—} (2)$$

Multiplying (1) by P_m and (2) by P_n

then subtracting, we have

$$P_m \cdot \frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} - P_n \cdot \frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\}.$$

$$+\{n(n+1) - m(m+1)\} P_n P_m = 0$$

Integrating between limits -1 to 1, we have

$$\int_{-1}^{+1} P_m \cdot \frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} dx - \int_{-1}^{+1} P_n \cdot \frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\} dx$$

$$+\{n(n+1) - m(m+1)\} \int_{-1}^{1} P_m P_n \, dx = 0$$

Integrating by parts,

$$\left[P_m (1-x^2)\frac{dP_n}{dx}\right]_{-1}^{+1} - \int_{-1}^{+1} \frac{dP_m}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} dx$$

$$-\left[P_n \cdot (1-x^2)\frac{dP_m}{dx}\right]_{-1}^{+1} - \int_{-1}^{+1} \frac{dP_n}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\} dx$$

$$+[n(n+1) - m(m+1)] \int_{-1}^{+1} P_m P_n \, dx = 0$$

$$\therefore \{n(n+1) - m(m-1)\} \int_{-1}^{+1} P_m P_n \, dx = 0$$

Hence $\displaystyle\int_{-1}^{+1} P_m(x) P_n \, dx = 0$ Since $m \neq n$.

(II) we have

$$(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$$

Squaring both sides, we have

$$(1-2xh+h^2)^{-1} \sum_{n=0}^{\infty} h^{2n} \{P_n(x)\}^2 + 2 \sum_{\substack{m,n=0 \\ m \neq n}} h^{m+n} P_m(x) P_n(x)$$

Integrating b/w limits -1 to $+1$, we have

$$\sum_{n=0}^{\infty} \int_{-1}^{+1} h^{2n} [P_n(x)]^2 dx + \sum_{\substack{m,n=0 \\ m \neq n}}^{\infty} \int_{-1}^{+1} h^{m+n} P_m(x) P_n(x) \, dx$$

$$= \int_{-1}^{+1} \frac{dx}{(1-2xh+h^2)}$$

$$\sum_{n=0}^{\infty} \int_{-1}^{+1} h^{2n} [P_n(x)]^2 dx = \int_{-1}^{+1} \frac{dx}{(1-2xh+h^2)}$$

Since other Integrals on the LHS are zero by ①

as $m \neq n$

$$= -\frac{1}{2h} \left\{ \log (1-2xh+h^2) \right\}_{-1}^{+1}$$

$$= \frac{-1}{2h} \left\{ \log (1-h)^2 - \log (1+h)^2 \right\}$$

$$= \frac{1}{2h} \left[\log \left\{ \frac{1+h}{1-h} \right\}^2 \right]$$

$$= \frac{1}{h} \log \left\{ \frac{1+h}{1-h} \right\}$$

$$= \frac{2}{h} \left[h + \frac{h^3}{3} + \frac{h^5}{5} + \cdots \right]$$

$$= 2 \left\{ 1 + \frac{h^2}{3} + \frac{h^4}{5} + \cdots + \frac{h^{2n}}{2n+1} + \cdots \right\}$$

$$= \sum_{n=0}^{\infty} \frac{2h^{2n}}{2n+1}$$

equalting coeff of h^{2n}, we have

$$\int_{-1}^{+1} [P_n(x)]^2 \, dx = \frac{2}{2n+1}$$